

Discussion Problems 1

Some of these questions from Maurizio Caligaris

Every week, we will provide a handout of discussion problems that we hope will help you get a better understanding of the concepts we're currently covering. We recommend stopping by the weekly problem sessions to work through these problems in a group and to get advice from the TAs. Although we won't collect these for a grade, you are **strongly encouraged** to work on them, since they'll help prepare you for the problem sets and will (hopefully!) elucidate some of the more nuanced points we've covered.

Problem One: Identity Elements

If \star is a binary operator over some set S , then an element $z \in S$ is called an *identity element* for \star if for every $a \in S$, the following identity holds:

$$a \star z = z \star a = a$$

Some binary operators have identity elements, and some do not. However, no binary operators have several different identity elements.

Suppose that \star is a binary operator over a set S . Prove that if \star has identity elements z_1 and z_2 , then $z_1 = z_2$. This is an example of a *uniqueness proof*, in which you show that at most one object with a certain property can exist by proving that if there are two objects with a certain description, they must actually be the same object.

Problem Two: Balls in Bins

Suppose that you have twenty-five balls to place into five different bins. Eleven of the balls are red, while the other fourteen are blue. Prove that no matter how the balls are placed into the bins, there must be at least one bin containing at least three red balls.

Problem Three: Russell's Paradox

Consider the following set R :

$$R = \{ S \mid S \text{ is a set and } S \notin S \}$$

Now consider the following question: Is R an element of itself?

Prove that the statement $R \in R$ is a paradox (the statement $R \in R$ leads to a contradiction regardless of whether it's true or false.)

Problem Four: Quadratic Equations

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$. A *root* of the equation is a real number x satisfying the equation.

Recall from lecture that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ± 1 .

- i. Prove that mn is odd iff m is odd and n is odd.
- ii. Prove, by contradiction, that if a , b , and c are odd numbers, then there are no rational numbers x for which $ax^2 + bx + c = 0$. As a hint, if the rational solution is p/q , consider what happens if both p and q are odd and what happens if exactly one of p and q is odd. (Why can't both p and q be even?)

Problem Five: Finding Flaws in Proofs

The following proofs all contain errors that allow them to prove results that are incorrect. For each proof, identify at least one flaw in the proof and explain what the problem is.

Theorem: If n is even, then n^2 is odd.

Proof: By contradiction; assume that n is odd but that n^2 is even. Since n is odd, $n = 2k + 1$ for some integer k . Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts our earlier claim that n^2 is even. We have reached a contradiction, so our initial assumption was wrong. Thus if n is even, n^2 is odd. ■

Theorem: For all sets A and B , we have $A \cup B = A$.

Proof: By contradiction; assume that for all sets A and B , we have $A \cup B \neq A$. Since this claim holds for any choice of sets A and B , in particular it should hold when $A = \emptyset$ and $B = \emptyset$. However, in this case, $A \cup B = \emptyset \cup \emptyset = \emptyset$, which is the set A . Therefore, $A \cup B = A$, contradicting our earlier claim that $A \cup B \neq A$ for all A and B . We have reached a contradiction, so our initial assumption was wrong. Therefore, for any sets A and B , we have $A \cup B = A$. ■

Theorem: If $C \subseteq A \cup B$, then $C \subseteq A$.

Proof: By contrapositive. We prove that if C is not a subset of $A \cup B$, then it is not a subset of A . Since C is not a subset of $A \cup B$, there is some $x \in C$ such that $x \notin A \cup B$. Since $x \notin A \cup B$, we know that $x \notin A$ and that $x \notin B$. Thus $x \in C$ but $x \notin A$, and so C is not a subset of A . ■